

Entanglement, number fluctuations and optimized interferometric phase measurement

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We derive a phase-entanglement criterion for two bosonic modes which is immune to number fluctuations, using the generalized Moore-Penrose inverse to normalize the phase-quadrature operator. We also obtain a phase-squeezing criterion that is immune to number fluctuations using similar techniques. These are utilized to obtain an operational definition of relative phase-measurement sensitivity, via analysis of phase measurement in interferometry. We show that these measures are proportional to enhanced phase-measurement sensitivity. The phase-entanglement criterion is a hallmark for a new type of quantum squeezing, namely planar quantum squeezing. This has the property that it squeezes two orthogonal spin directions simultaneously, which is possible owing to the fact that the $SU(2)$ group that describes spin symmetry has a three-dimensional parameter space, of higher dimension than the group for photonic quadratures. The practical advantage of planar quantum squeezing is that, unlike conventional spin-squeezing, it allows noise reduction over all phase-angles simultaneously. The application of this type of squeezing is to quantum measurement of an unknown phase. We show that a completely unknown phase requires two orthogonal measurements, and that with planar quantum squeezing it is possible to reduce the measurement uncertainty independently of the unknown phase value. This is a different type of squeezing to the usual spin-squeezing interferometric criterion, which is only applicable when the measured phase is already known to a good approximation, or can be measured iteratively. As an example, we calculate the phase-entanglement of the ground state of a two-well, coupled Bose-Einstein condensate, similar to recent experiments. This system demonstrates planar squeezing in both the attractive and repulsive interaction regimes.

I. INTRODUCTION

Entanglement criteria are widely used to identify non-classical resources for potential applications in quantum technology. One application is the enhancement of measurement sensitivity. In practice, the most sensitive measurements are often interferometric. Hence, the measurement of an unknown quantity is reduced to the measurement of a phase-shift. In this communication, we analyze how non-classical, entangled states can increase phase-measurement sensitivity. To achieve this, we will introduce both an operational measure of relative phase, and a corresponding signature of phase entanglement between two Bose fields, using well-defined interferometric particle-counting procedures. This is shown to quantitatively measure the enhancement of an interferometric measurement. It is the interferometric equivalent of the spin-squeezing criterion [1–3], which is known to measure the non-classical precision of a clock [3].

We introduce a relative phase operator which is well-defined in the case of variable total particle number, by using the generalized inverse method to prevent singularities in the inverse number operator. It is the simplest relative phase operator that is measurable interferometrically. Number fluctuations are always found

in current experimental photonic and atom interferometer phase measurements. Hence, we clarify the operational phase-measurement procedure already used heuristically to analyze experiments [4–7]. Most importantly, we show that when the measured phase is unknown prior to measurement, a state preparation that involves mode-entanglement is optimal, and is closely related to the relative-phase operator. This complements previous studies which generally assume either that the phase is already known to a good approximation, or that the phase-shift remains constant during repeated measurements. Here, we use the minimal number of measurements possible. This is inherently different to strategies employed to estimate phase through sequential or multiple measurements, which assume the phase-shift is a classical, time-invariant quantity.

Our entanglement measure is a normalized form of the recently introduced Hillery-Zubairy (HZ) non-hermitian operator product criterion [8], similar to that introduced in a previous paper [9]. We prove that this normalized form is a phase-entanglement signature for two Bose fields, and has the advantage of being almost immune to total number fluctuations. We show how this criterion can be interpreted as a variance measure which quantifies entanglement, and has a direct physical interpretation as the enhancement of phase measurement sensitivity in an interferometer. This is directly related to the idea of planar quantum squeezing, in which the quantum noise is simultaneously reduced in two orthogonal directions of phase-measurement.

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The present analysis focuses on linear, two-mode interferometry with particle-counting detectors. This technique is by far the most commonly used technique for phase-measurement, and therefore deserving of a careful analysis. Our results, while less general than an analysis of completely arbitrary quantum measurements, are able to be implemented immediately, since two-mode interferometers are widely available both for photonic and atomic fields. We discuss techniques for generating the required entangled input fields using atomic sources. As an example, we consider how to obtain these types of quantum state through the creation of a correlated ground-state in a coupled, two-mode Bose-Einstein condensate (BEC) with either attractive or repulsive S-wave scattering interactions. We show that as well as giving sub-shot noise (squeezed) phase noise in two orthogonal phase directions simultaneously, it is possible to obtain nearly Heisenberg-limited performance in one of two phase directions, which is useful when the phase is known approximately.

II. QUANTUM PHASE MEASUREMENTS

Interferometers are designed to measure a relative phase-shift ϕ , typically between two beams. The phase information must first be encoded on quantum fields before it is measured. Hence, there is a close relationship between interferometry, which measures a phase-shift in a medium, and the measurement of phase of a quantum field, which is where the interferometric phase information is stored. The relationship is more important than meets the eye. An experimentally measured phase-shift is can neither be truly classical nor time-invariant, which are commonly used assumptions. For a variety of reasons, repeated measurement is not always possible, and one must regard interferometric measurement as primarily a quantum measurement problem.

In view of this, we first review earlier approaches to phase measurement of quantum fields, in order to explain and motivate the approach used in the rest of the paper.

A. Phase operators

We start with a generic problem in quantum phase measurement: how can one measure the phase of a quantum field or mode (\hat{a}). Classically, one divides up a field amplitude into intensity (N_a) and phase (ϕ_a) by introducing:

$$a = \sqrt{N_a} e^{i\phi_a}. \quad (1)$$

Next, if one measures the complex amplitude a , one simply classically normalizes to obtain the classical phase, $\phi_a = -i \ln [a/\sqrt{N_a}]$. We note that this gedanken-experiment for classical phase measurement involves measuring both real and imaginary components of the amplitude.

Quantum studies of this generic problem date back to the early attempt of Dirac [10] to define a quantum phase operator from a canonical commutation relation of form $[\hat{\phi}_a, \hat{N}_a] = -i$. The underlying mathematical problem is that a hermitian quantum operator $\hat{\phi}_a$ for the phase of a single harmonic oscillator operator \hat{a} strictly does not exist, which has been the topic of many previous studies [11–13]. This can be seen most easily from the commutation relations, $[\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}] = 1$. If phase is hermitian, then $\hat{a} = \sqrt{\hat{N}}\hat{U}$, where \hat{U} is a unitary operator such that $\hat{U}^\dagger\hat{U} = \hat{U}\hat{U}^\dagger = 1$, and $\hat{N} = \hat{a}^\dagger\hat{a}$. From the commutators, $\hat{N} - 1 = \hat{U}\hat{N}\hat{U}^\dagger$. This violates unitarity, since a unitary transformation cannot change an operator's eigenvalues.

This problem does not of course occur classically, and it is at the root of quantum phase-measurement problems. However, the idea of a phase-operator with canonical commutators is approximately valid at large particle number, and suggests the existence of a fundamental uncertainty principle called the Heisenberg limit:

$$\Delta\phi\Delta N \geq \frac{1}{2}.$$

Given that $\Delta N \leq N_{max}/2$, this has led to the idea of a Heisenberg limit of $\Delta\phi \geq 1/N_{max}$ on phase-measurement with at most N_{max} particles. Sometimes one interchanges \bar{N} and N_{max} , as they are often related.

B. Truncated Hilbert space methods

One resolution of the lack of a hermitian phase operator due to Pegg and Barnett, is to truncate the Hilbert space to a maximum boson number of s [14]. Next, one defines a phase eigenstate $|\theta\rangle_p$ as a discrete Fourier transform of number states $|n\rangle$, using:

$$|\theta\rangle_p = \frac{1}{\sqrt{s+1}} \sum_{n=0}^s e^{in\theta} |n\rangle. \quad (2)$$

From this starting point, it is clear that a hermitian phase operator can simply be obtained from the definition:

$$\hat{\phi}_a = \sum_{m=0}^s \theta_m |\theta_m\rangle \langle \theta_m|. \quad (3)$$

Here $\theta_m = \theta_0 + 2\pi m/(s+1)$, and θ_0 is a reference phase. This is a mathematically consistent approach, which resolves the issue of hermiticity, but it leaves a number of practical questions unanswered.

In particular, what is the physical meaning of the maximum number s and reference phase θ_0 ? Is it possible to take the limit of large s in a unique way? From an operational perspective, what (if any) is the relationship between the abstract operator $\hat{\phi}_a$ and an interferometric measurement? One of the purposes of this paper is to understand how these questions can be answered and implemented using interferometry.

C. Relative phase operator

Another solution along these directions is to define a relative phase operator, $\hat{\phi} = \hat{\phi}_a - \hat{\phi}_b$, for two modes a and b [15]. This has the conceptual advantage that it corresponds to operational procedures - which always involve relative phase measurement - and clarifies the meaning of the reference phase. A relative phase operator is consistent philosophically with the fundamental idea of relative measurements in physics, and is the most operationally meaningful way to define quantum phase in many cases.

With this approach, one works in a space whose algebra is defined by the equivalent angular momentum operators in the Schwinger representation:

$$\begin{aligned} \hat{J}^X &= \frac{1}{2} (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger), \\ \hat{J}^Y &= \frac{1}{2i} (\hat{a}^\dagger \hat{b} - \hat{a} \hat{b}^\dagger), \\ \hat{J}^Z &= \frac{1}{2} (\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}), \\ \hat{N} &= \hat{a}^\dagger \hat{a} + \hat{b}^\dagger \hat{b}. \end{aligned} \quad (4)$$

It is usual to assume that one has an eigenstate of the total number \hat{N} , with eigenvalue N and hence an equivalent angular momentum eigenstate of $J = N/2$. As we show later, this assumption is questionable in real experimental measurements. The states of well-defined phase then correspond to linear combinations of angular momentum eigenstates with $\hat{J}^Z |J, m\rangle = m |J, m\rangle$, and one then has a physical phase basis of:

$$|\theta\rangle_p = \frac{1}{\sqrt{2J+1}} \sum_{m=-J}^J e^{im\theta} |J, m\rangle. \quad (5)$$

This allows the definition of a relative phase-difference operator which is Hermitian, and has a discrete spectrum. The fact that the angular momentum Hilbert space is finite provides a natural explanation of the truncation parameter s in the Pegg-Barnett approach. However, it is still not immediately clear how to relate this abstract proposal to interferometric phase-measurements.

We also note that in practical interferometry experiments, it is nearly impossible have an input state that has a well-defined total particle number, especially at large mean particle number. Instead, the most common situation is that there is an initial mixture of particle numbers, with number fluctuations that are typically at least Poissonian. We will return to the issues of interferometric measurement and number fluctuations in later sections.

D. Quantum sine and cosine operators

An alternative resolution of the phase-measurement question is to define quantum sine and cosine operators. This is a way to reach the phase through measurement

of the real or imaginary part of the amplitude, an idea which has a clear analog in the classical world, as described above. Operationally, this is the quantum version of a proposal by Zernike [16] to analyze coherence in classical interferometry. The original idea of Zernike was to relate coherence properties directly to measured classical fringe visibility.

Extending this to the quantum theory of a single quantized mode [11], one can define a normalized amplitude:

$$\hat{E} = [1 + \hat{N}]^{-1/2} \hat{a}, \quad (6)$$

from which the sine and cosine operators are obtained via:

$$\begin{aligned} \hat{C} &= \frac{1}{2} [\hat{E} + \hat{E}^\dagger], \\ \hat{S} &= \frac{1}{2i} [\hat{E} - \hat{E}^\dagger]. \end{aligned} \quad (7)$$

This is still operationally unclear for general applications, except in the limit where one arm of an interferometer is a large ‘classical-like’ local oscillator. In this limit, the approach is closely related to the theory of optical quadratures, where the two operators involved have commutators similar to the quantum position and momentum operators: for a field mode \hat{a} , the quadrature amplitudes are defined $\hat{X} = (\hat{a} + \hat{a}^\dagger)/2$, and $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(2i)$.

A drawback of the local oscillator approach implicit in this method is that in practical terms it is highly resource-hungry. A classical local oscillator must necessarily involve a very large boson number in the local oscillator mode, even though this mode is not formally included as part of the measured system. We note that there is a generic issue, which is that since these operators do not commute, it is not possible to measure both quadratures simultaneously.

E. Relative phase quadrature operators

Alternatively, the same general idea of quadrature measurement can be applied to the relative phase between two modes. This also allows a better understanding of the actual resources involved in the measurement, since it effectively combines both the measured beam and the local oscillator into the measured operator. The combined approach is studied in an early review of Carruthers and Nieto [13], and reduces to defining the cosine and sine operators of the phase-difference as:

$$\begin{aligned} \hat{C}_{12} &= \hat{C}_1 \hat{C}_2 + \hat{S}_1 \hat{S}_2, \\ \hat{S}_{12} &= \hat{S}_1 \hat{C}_2 - \hat{S}_2 \hat{C}_1. \end{aligned} \quad (8)$$

This has the virtue that these operators commute with the total particle number, $\hat{N} = \hat{N}_1 + \hat{N}_2$. This number is kept finite, and provides an indication of the true resource needed for the measurement. A related definition for a

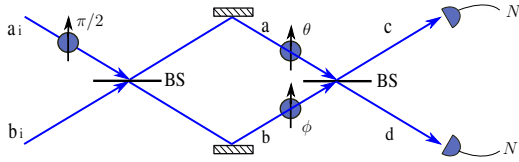


Figure 1. Interferometric measurements can be reduced in the simplest case to a measurement of outputs from a beamsplitter.

two-mode BEC was proposed by Leggett [17], who suggested defining a phase operator following the approach of Carruthers and Nieto, except that:

$$\hat{E} = \left[\left(N/2 - \hat{J}^Z \right) \left(N/2 + \hat{J}^Z + 1 \right) \right]^{-1/2} \left(\hat{J}^X + i\hat{J}^Y \right). \quad (9)$$

Here the phase is encoded in the rotations in the $X-Y$ plane, a convention we will follow in this paper unless otherwise noted. In these two approaches, the proposed quadrature operators were not analyzed from the perspective of interferometric fringes and their noise properties. We will show that the operationally most relevant approach to interferometry measurements is to use a different normalization to either of the above suggestions.

III. PHASE OPERATORS AND INTERFEROMETRY

An operational analysis of interferometric measurements can be reduced in the simplest case to a measurement of outputs from the final beam-splitter as shown in figure 1, with one mode b experiencing an unknown phase-shift of ϕ , while the other mode a is shifted by a fixed reference phase θ . The other components of the interferometer are then part of the quantum state preparation that determines the output expectation values. This is shown schematically as the input beam-splitter with input modes a_i, b_i , in the schematic Mach-Zehnder (MZ) diagram in figure 1.

In the following, we shall mostly focus on the general scheme in which the intermediate modes a, b have an arbitrary state preparation. However, we shall also treat particular examples where the state preparation is obtained through the Mach-Zehnder protocol. In this case we analyze the state preparation of modes a_i, b_i , as a practical route towards preparing the intermediate modes, and also introduce an additional phase-shift on the input to the MZ, for reasons explained in the last section.

A. Quantum limits to classical phase-estimation

The problem of measuring a quantum phase $\hat{\phi}$ is related, but not identical, to the analysis of quantum limits to estimation of a classical phase. There is an essential difference, since classical phase-estimation usually

assumes there is a phase-shifting element that produces a well-defined classical phase-shift ϕ , which is supposed to be time-independent. Many treatments assume that the phase can be measured iteratively without disturbing it, to improve the accuracy. This limitation rules out many situations where the phase evolves in time - or where the phase experiences a back-action which changes the phase after measurement. Other approaches to the problem make the assumption that it is possible to construct arbitrary quantum states and measuring devices. As a result, these treatments generally are not applicable to two-mode interferometry, although they may be applicable to some future phase-measuring device.

We first introduce earlier approaches to phase-estimation. Pioneering work by Caves [18], who treated bosons in the context of gravity-wave detection, showed that two-mode interferometry sensitivity could be improved above the shot-noise level i.e., the *Standard Quantum Limit* (SQL) of $\Delta\phi = 1/\sqrt{N}$:

$$\Delta\phi < 1/\sqrt{N} \quad (10)$$

This required non-classical, ‘squeezed state’ input radiation, reaching a maximum sensitivity near the *Heisenberg limit* of

$$\Delta\phi = 1/\bar{N} \quad (11)$$

for an input state with \bar{N} average particle number. Squeezed states allow the uncertainty of one observable to be reduced below the standard quantum limit, at the expense of the complementary observables, so that the Heisenberg uncertainty relation is still satisfied [19]. Thus, for single mode optical amplitudes where $\hat{X} = (\hat{a} + \hat{a}^\dagger)/2$, and $\hat{P} = (\hat{a} - \hat{a}^\dagger)/(2i)$, for which $\Delta\hat{X}\Delta\hat{P} \geq 1/4$, squeezing of \hat{X} occurs when $\Delta\hat{X} < 1/2$. This is clearly very closely related to the quadrature phase-operator [11] approach of equation (7).

Paradoxically, the usual squeezed state technologies of parametric down-conversion are rather inefficient. The total resources employed, in terms of boson number prior to down-conversion, are generally no better than coherent interferometry[20] for a given phase sensitivity. As pointed out by Caves, there is still an advantage of this method for gravity-wave detectors, as it reduces the back-action caused by radiation pressure.

The treatment of Caves also assumed prior knowledge of the unknown phase, to allow a linearized treatment in the limit of small fluctuations. This type of analysis was applied to fermion interferometry by Yurke [21, 22], and was later extended to multiple measurements [23], with similar conclusions, except for the replacement of \bar{N} by the total number of particles involved, \bar{N}_{TOT} .

Squeezed quantum fluctuations have been shown to enhance the sensitivity of other sorts of measurements [24, 25]. Wineland et. al., in the context of atomic clock measurements [3], showed that there was a close relationship between interferometry and the concept of spin-squeezing [1]. The uncertainty relation for spin is

$\Delta\hat{J}^Y\Delta\hat{J}^Z \geq |\langle\hat{J}^X\rangle|/2$, and spin squeezing exists when the variance of one spin is reduced below the standard quantum limit (SQL) [1]:

$$\Delta\hat{J}^Y < \sqrt{|\langle\hat{J}^X\rangle|/2}. \quad (12)$$

The spin squeezing factor

$$\xi_S = \frac{\sqrt{2J}\Delta\hat{J}^Y}{|\langle\hat{J}^X\rangle|} \quad (13)$$

was introduced for a collection of N two-level atoms or equivalently, for two occupied modes for which a collective pseudo-spin is defined, and $J = N/2$. In spectroscopy or interferometry, the final measurement is that of a spin component, measured as a population difference between the two quantum states.

Since we wish to consider phase in the initial state prior to a beamsplitter, we consider that the x direction is chosen to be that of the large spin vector, so that $\langle\hat{J}^X\rangle \sim J$. Fluctuations are thus spin-squeezed when $\Delta^2\hat{J}^Y < J/2 = N/4$. The precision of the quantum measurement is given by [3, 26]

$$\Delta\phi = \xi_S/\sqrt{N} \quad (14)$$

which is enhanced over the SQL (10) when

$$\xi_S < 1 \quad (15)$$

and reaches the Heisenberg limit as $\xi_S \rightarrow 1/\sqrt{N}$. This implies that there are large fluctuations in \hat{J}^Z . The most extreme case of this is when we choose the input state to be the eigenstate of the two-mode phase operator, Eq (5), which establishes a connection between these two approaches. In this case, one finds that

$$\Delta^2\hat{J}^Z = \frac{N}{12} [N + 2]. \quad (16)$$

Further analysis has suggested ways to reach these Heisenberg limits (11) through macroscopic superpositions or ‘N00N’ states [26]. More recently, nonlinear interferometry [27] was suggested as a route to go beyond the Heisenberg limit, although this requires specific nonlinear couplings. We note that all of these techniques use a linearized approach to phase-estimation. This means that the phase must already be known to an excellent approximation prior to measurement. In practice, this implies repeated measurements to refine the estimation of the phase until the final, high-precision measurement is made. In particular, we show below that a two-mode phase eigenstate only gives a low interferometric measurement variance when the phase is known almost perfectly prior to the measurement.

There are general treatments that typically involve more than single-pass, two-mode interferometry [28]. Sanders and Milburn [29] found the optimal measurement and state to determine the phase ϕ , based on the two-mode phase-operator approach of equation (5). Berry

and Wiseman [30, 31] showed that this canonical measurement cannot be realized by counting particles in an interferometer (figure 1), and proposed alternative iterative schemes. Such idealized measurements have also been analyzed by many authors using techniques like the quantum Fisher information [32, 33].

The assumption of a time-invariant, classical phase-shift used in iterative schemes can rule out such techniques for many applications. Phases often vary in time, and one cannot always avoid quantum back-action. The physical reason for this is that a phase-shift corresponds to an energy-shift term in the radiation-matter Hamiltonian [34]. This therefore has dynamical consequences for the object being measured, and may change the phase. Hence, the use of iterative or repeated measurement schemes is not always feasible. Strategies of this type are different to the present theory, which focuses on minimal numbers of measurements using a two-mode linear interferometer.

B. Phase-measurement operator

We now return to the fundamental question of how to measure a phase-shift using a quantum phase-measurement operator. We wish to use a strictly operational definition, solely utilizing the interferometer outputs. If we define the operator phases so that $c, d = [ae^{-i\theta} \pm be^{-i\phi}]/\sqrt{2}$, then the measured outputs of the quantum interferometer in terms of a, b are:

$$\begin{aligned} \hat{N}_{\pm} &= \hat{c}^\dagger c \pm \hat{d}^\dagger d \\ &= \frac{1}{2}\hat{N} \pm \frac{1}{2} [\hat{a}^\dagger \hat{b} e^{-i(\phi-\theta)} + \hat{b}^\dagger \hat{a} e^{i(\phi-\theta)}]. \end{aligned} \quad (17)$$

Comparing these quantities with the equivalent angular momentum operator approach from equation (4), we see that these quantities can be rewritten as: $\hat{N}_{\pm} = \hat{N}/2 \pm \hat{J}^\phi$, where \hat{N} is the total number operator, and:

$$\hat{J}^\phi = \hat{J}^X \cos(\phi - \theta) + \hat{J}^Y \sin(\phi - \theta).$$

In any given measurement, there are always two outputs which can be measured simultaneously. The quantum phase of the output field - which is also related to the unknown phase-shift ϕ - can be estimated from the ratio:

$$\tilde{J}(\phi) = \lim_{\epsilon \rightarrow 0^+} \frac{\hat{N}(\hat{N}_+ - \hat{N}_-)}{2(\epsilon + \hat{N}^2)} = \hat{J}^\phi \hat{N}^+. \quad (18)$$

Here \hat{N}^+ is the Moore-Penrose generalized inverse [35] of \hat{N} . This is a well-defined hermitian observable that commutes with \hat{J}^ϕ , and gives the least-squares solution of any inversion or variational problem involving \hat{N} . The generalized inverse \hat{N}^+ has many of the properties of a standard inverse, including the property that its eigenvalues are N^{-1} for number states with total number $N > 0$. However, it is zero (not infinity) for the vacuum state.

This means that, unlike the standard inverse, it has a well-defined value for *all* quantum states. More details are in the Appendix.

In experiments designed for accurate phase measurement for large average particle numbers, events with zero total particle number occur with vanishingly small probability, and generally can be neglected, in which case $\hat{N}^+ \approx \hat{N}^{-1}$. Of course, this is only true approximately, as the number operator has no standard inverse. The quantity $\tilde{J}(\phi)$ is always measurable and hermitian. Hence, $\tilde{J}(\phi)$ can be called a relative phase quadrature operator, and is the fundamental quantity measured in any phase-sensitive interferometric experiment.

It is vital to normalize by the particle number at each measurement - as indicated in the above operator - for the simple reason that in general, the particle number is not known in advance. It is often theoretically assumed that the total particle number is known prior to measurement. This is rarely found in real experiments, especially as the number is increased. An inspection of the experimental protocols actually used in recent BEC interferometry experiments [4, 6, 7] shows that the operator given above corresponds rather closely to the way that data is analyzed in practice. Our analysis therefore provides a theoretical justification for these operational procedures. We will treat the effects of typical Poissonian particle number fluctuations in a later section.

The operator $\tilde{J}(\phi)$ is different to both the complex phase-difference quadrature operators of Nieto and Carruthers [13], and of Leggett, owing to the type of normalization chosen here. If we compare the current approach of equation (18) with these earlier suggestions in equations (8) and (9), there is a very important difference. The operator $\tilde{J}(\phi)$ given above is uniquely defined for all inputs, and can be measured completely from a single, combined measurement of the two interferometer outputs. It is not obvious how one can measure the earlier proposed phase-measurement operators in practical interferometry experiments, since they appear to require the simultaneous measurement of non-commuting output operators like \hat{J}^X and \hat{J}^Z . Of course, this does not rule out more sophisticated operational measurements, as the combined operators are hermitian; but these measurements do not appear feasible with simple beam-splitters and photodetectors.

If multiple measurements are made sequentially, then more sophisticated iterative phase-estimation techniques are possible [23, 28, 30, 36, 37]. As pointed out above, this does not help in experiments where the phase is changing. Often, only a single measurement is possible. We also recall that many previous analyses are conditioned on having *a priori* approximate knowledge of the unknown phase-shift ϕ . In the following, we focus instead on optimizing the sensitivity of the operational phase measurement equation (18), for a range of unknown phase-shifts. In other words, we assume that the phase is known to lie in a given interval that is *not* vanishingly small.

C. Entanglement and squeezing

Interferometric sensitivity and particle entanglement have previously been linked through criteria involving Fisher information [32]. Sorenson et. al. [38] have shown that a measure of particle entanglement is the spin squeezing criterion (15): a fixed number N of two-level systems (spin-1/2 particles) are separable when $\rho = \sum_R P_R \rho_1^R \dots \rho_N^R$ and hence entangled when

$$0 < \xi_S^{Z/Y} = \frac{\sqrt{N} \Delta \hat{J}^{Z/Y}}{|\langle \hat{J}^X \rangle|} < 1. \quad (19)$$

Here we define the collective spins associated with N spin 1/2 systems: $\hat{J}^\theta = \sum_{k=1}^N \hat{J}_k^\theta$ ($\theta = X, Y, Z$), and \hat{J}_k^θ is the spin of the k -th particle. The Heisenberg Uncertainty Principle places a lower bound on ξ_S , because of the finite size of the Hilbert space. The precise values for the lower bound for fixed N have been determined by Sorenson and Molmer in Ref. [39], and decrease with increasing spin J . The spin squeezing criterion has been measured experimentally in BEC interferometry [4–6], and is related to phase-measurement efficiency when the phase value is approximately known in advance [3], as summarized by (14).

In this paper, we will generalize this criterion to include number fluctuations, and also treat a very different type of entanglement, that between two distinct spatially separated locations, rather than between many qubits. We will relate the sensitivity of the phase measurement (18) of an unknown phase to a special type of two-mode entanglement between a and b . Two-mode entanglement is defined as a failure of the separable model

$$\rho = \sum_R P_R \rho_a^R \rho_b^R \quad (20)$$

where $P_R > 0$, $\sum_R P_R = 1$ and $\rho_{a/b}^R$ are density operators for states at a/b .

Many criteria for two-mode entanglement exist, but we are interested only in those interferometric measures of entanglement that can enhance the phase measurement task of Figure 1 [32, 40, 41]. The observables that can be measured are given in equation (4) and (18). These expressions are written in terms of the modes a and b that are the inputs to the final beam splitter. One may also consider a MZ type of experiment with a phase-rotation and two beam-splitters as is illustrated in Figure 1. In this case, the MZ internal modes are related to the input modes a_i, b_i by $a = (-ia_i + b_i)/\sqrt{2}$, $b = -(ia_i + b_i)/\sqrt{2}$, and the measured outputs are simply rotated versions of the Mach-Zehnder input operators, with:

$$\begin{aligned} \hat{J}^X &= \hat{J}_i^Z, \\ \hat{J}^Y &= -\hat{J}_i^X. \end{aligned} \quad (21)$$

We see from these expressions (4) and (18) that the sensitivity of the phase measurement will depend on the

noise levels of the two orthogonal components, $\hat{J}^X \hat{N}^+$ and $\hat{J}^Y \hat{N}^+$ (in terms of the interferometer modes a and b). It makes sense to then choose an input state for the interferometer that will maximally reduce the noise in *both* of these components simultaneously. In fact this requirement is closely related to an entanglement measure. Hillery and Zubairy (HZ) showed [8] that for any *separable* state (20),

$$\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y \geq \langle \hat{N} \rangle / 2. \quad (22)$$

Entanglement between modes a and b is thus detected when (22) fails:

$$0 < E_{HZ} = \frac{\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y}{\langle \hat{N} \rangle / 2} < 1. \quad (23)$$

$E_{HZ} = 1$ gives the Standard Quantum Limit (SQL) noise level, which is the noise level E_{HZ} obtained when the modes a and b are in the separable product of coherent states, $|\alpha\rangle|\beta\rangle$. It is not possible however to choose a state so that both variances $\Delta^2 \hat{J}^X$, $\Delta^2 \hat{J}^Y$ are zero.

D. Planar Quantum Squeezing

If we consider the Heisenberg uncertainty principle in the $X-Y$ plane, we see that it has the form $\Delta \hat{J}^Y \Delta \hat{J}^X \geq |\langle \hat{J}^Z \rangle|/2$. Here the optimal situation is obtained for equal beam intensities entering the beam-splitter, so that $\langle \hat{J}^Z \rangle = 0$. This appears to provide no lower bound to the measured quadrature variances, and hence to the phase noise. However, appearances can be very misleading. In fact, a non-zero lower bound to E_{HZ} exists, because the variances of \hat{J}^X and \hat{J}^Y cannot be simultaneously zero.

For fixed $N = 2J$, this bound has been determined. It is known that

$$C_J/J \leq E_{HZ}, \quad (24)$$

where the coefficients $C_J \sim 3(2J)^{2/3}/8$ as $J \rightarrow \infty$ [43]. This means, however, that both the orthogonal variances in a phase-measurement can be simultaneously reduced below the shot-noise level, since we are minimizing the sum of the phase variances, $\Delta^2 \hat{J}^X + \Delta^2 \hat{J}^Y$. In general, a noise reduction of the sum of two variances below the shot-noise level is called planar quantum squeezing or PQS [43], as it minimizes quantum noise in a plane, rather than just in one direction on the Bloch sphere. It has the advantage that noise reduction for phase measurement occurs regardless of the value of the unknown phase.

It is instructive to compare the optimal PQS state with the relative phase eigenstate. The minimal variance PQS state is:

$$|\psi\rangle = \frac{1}{\sqrt{2J+1}} \sum_{m=-J}^J R_m e^{im\theta} |J, m\rangle. \quad (25)$$

The asymptotic limit of the optimal coefficient R_m , which minimizes the sum of the quadrature variances, is then a Gaussian of form:

$$R_m = \frac{e^{-m^2/(2\sigma_m)}}{\sqrt{2\pi\sigma_m}}. \quad (26)$$

where the variance in the space of \hat{J}^Z eigenvalues is $\sigma_m = \Delta^2 \hat{J}^Z = (J^2/2)^{2/3}$. The other important properties are that, to leading order: $\langle \hat{J}^X \rangle \sim J$, $\Delta^2 \hat{J}^X \sim (2J)^{2/3}/8$, $\Delta^2 \hat{J}^Y \sim (2J)^{2/3}/4$. Thus, the optimal PQS state reduces the variance in \hat{J}^X and \hat{J}^Y simultaneously, with *both* variances well below the shot-noise level.

By contrast, for a relative phase state at large J , we have $\langle \hat{J}^X \rangle \sim \pi J/4$, $\Delta^2 \hat{J}^X \sim (2/3 - \pi^2/16)J^2$, $\Delta^2 \hat{J}^Y \sim \sqrt{\ln(J)}$, $\Delta^2 \hat{J}^Z \sim J^2/3$. Hence, the Heisenberg uncertainty principle in the $Z-Y$ plane is obeyed, since $\Delta \hat{J}^Y \Delta \hat{J}^Z > |\langle \hat{J}^X \rangle|/2$. However, quantum squeezing only occurs in the Y spin direction, while in both the other spin directions the noise is greatly increased *above* the shot-noise level. This means that in an interferometric measurement using a relative phase state, the reference phase-offset θ must be adjusted to match ϕ with high precision, even though ϕ is of course unknown prior to measurement. This adjustment is necessary to avoid contamination of the results with high levels of noise from the X spin direction, which are well above the shot-noise or Poissonian level. The underlying cause is that interferometric measurements do *not* simply project out the relative phase eigenstates.

We see that the main advantage of PQS states in interferometry is that it is possible to have sub-shot precision in both the measured spin directions simultaneously. This is advantageous when the measured phase is truly unknown. At first, it may seem that this is less than optimal as a squeezing strategy when the phase is known approximately. For the optimal PQS state described above, which minimizes the variance sum, neither of the variances are close to the Heisenberg limit. Importantly, there are a range of possible PQS states in which the relative variances in the X and Y directions can be adapted to the desired measurement strategy, including states in which PQS - with the variance sum below the shot-noise level - is combined with nearly Heisenberg-limited variance reduction in one of the two directions. This possibility is discussed in the last section, together with practical techniques for achieving it.

IV. OPERATIONAL CRITERIA AND NUMBER FLUCTUATIONS

In practical interferometry, the total number of input bosons usually changes at each measurement. Hence the ensemble used for averaging has a finite distribution over the particle number. This is caused by a number of factors. In optical lasers, it is caused by technical noise

in the optical pumping process, as well as well-known quantum noise effects during stimulated emission and out-coupling [44]. In BEC and atom lasers, the factors involved range from fluctuations in the initial atomic density distribution in the magneto-optical trap, to quantum noise due to the atomic collisions that occur in the evaporative cooling process [45]. These number fluctuations are due to the non-equilibrium mechanisms that generate a laser or BEC respectively, and there is no reason to assume either a canonical or a grand canonical ensemble.

The direct use of the HZ spatial entanglement criterion is highly sensitive to total number fluctuations. For this reason, we will introduce entanglement and spin-squeezing definitions that are normalized by the total number. We refer to such normalized entanglement measures as *phase-entanglement* and *phase-squeezing* measures, as they measure correlated and reduced noise phases. There is another possible strategy, which is to simply reject all measurements that have the ‘wrong’ particle number. This allows a conditional number state measurement to be obtained. While this is feasible, it is also extremely inefficient, since most attempted measurements yield no information at all about the phase with this strategy.

A. Experimental number fluctuations

In the case where the number fluctuations are Poissonian, the probability that there are exactly N bosons is

$$P(N) = \frac{1}{N!} \langle \hat{N} \rangle^N e^{-\langle \hat{N} \rangle}. \quad (27)$$

While this may not be the best input state for a phase measurement, this distribution does give a number standard deviation of $\sigma_N = \sqrt{N}$, which is a typical order of magnitude for the number fluctuations in a well-stabilized photonic laser or BEC. This fluctuation is in the total number N , prior to any beam-splitter. Interferometric beam-splitters introduce relative number fluctuations in addition to the total number fluctuation.

Highly-stabilized semiconductor lasers have reached slightly lower number variances than Poissonian, in a restricted frequency range [46]. For an atomic BEC experiment, atom number statistics are difficult to measure accurately to this level of precision for large \bar{N} . Number fluctuations of at least the Poissonian level are found in almost all current BEC experiments [6] where data is available. Just as with lasers, it is possible to obtain lower variances than this, with some restrictions. In the best results to date, standard deviations as low as $0.6\sqrt{\bar{N}}$ (below the Poissonian level) were observed at very small atom numbers of $\bar{N} \approx 60$. Super-Poissonian variances were found for larger numbers of $N > 500$ [47].

When calculating the entanglement parameter E_{HZ} for typical states with Poissonian fluctuations, we find that the apparent entanglement is greatly decreased due

to the effect of total number fluctuations. This is misleading: total number fluctuations do not destroy entanglement. Accordingly, it is important to use entanglement and phase-sensitivity measures that allow for number fluctuations. Our general criteria therefore include number fluctuations with an arbitrary variance. These criteria can still be used to describe idealized experiments without number fluctuations, even though this is not very realistic. In the next section, we will make use of a Poissonian distribution to model the behavior of typical BEC experiments, with a low atom number of $\bar{N} \sim 100$.

B. Entanglement and squeezing criterion

To treat phase measurement including total number fluctuations, we have introduced normalized spin operators: $\tilde{J}^\theta = \hat{J}^\theta \hat{N}^+$. Here, \hat{N}^+ is the Moore-Penrose generalized inverse of the number operator. Detailed properties and proofs are given in the Appendix. We can now use these normalized operators to derive general operational criteria for entanglement and squeezing, which extend the results obtained above to a realistic environment with number-fluctuations. We note that somewhat different results have been obtained previously with number fluctuations, in work that used un-normalized operators [48].

1. Phase-entanglement criterion

We now introduce a phase-entanglement measure that is robust against number fluctuations. We show in the Appendix that in a number-fluctuating environment, entanglement between modes a and b is confirmed via a phase-entanglement criterion that uses the generalized Moore-Penrose inverse of the number operator, \hat{N}^+ :

$$E_{ph} = \frac{\Delta^2 \tilde{J}^X + \Delta^2 \tilde{J}^Y}{\langle \hat{N}^+ \rangle / 2} < 1. \quad (28)$$

2. Phase-squeezing criterion

Similarly, we show in the Appendix that entanglement between N spin-1/2 systems is confirmed by a normalized spin squeezing criterion, which we term phase-squeezing:

$$\xi_{S,ph}^{Z/Y} = \frac{\sqrt{\langle \hat{N} \rangle} \Delta \tilde{J}^{Z,Y}}{|\langle \tilde{J}^X \rangle|} < 1. \quad (29)$$

The two criteria (28) and (29) and their application to determine the enhanced sensitivity of a two-mode atom interferometer, in particular a BEC atom interferometer for which incoming number fluctuations are included, form the major results of this paper.

C. Phase sensitivity

Next, we will obtain a detailed understanding of the relationship between our phase-entanglement measure, and phase-measurement sensitivity. The crucial issue in phase measurement is the measurement sensitivity, or smallest measurable phase-shift. This is related to the differential signal to noise ratio, given by [21, 22]

$$\frac{dS}{d\phi} \equiv (\Delta\phi)^{-1} = \frac{1}{\sqrt{(\Delta\tilde{J})^2}} \left| \frac{d\langle\tilde{J}\rangle}{d\phi} \right|. \quad (30)$$

The smallest measurable change in phase in a single measurement is $\Delta\phi$. Figure 1 depicts an unknown phase shift ϕ (to be measured) relative to a fixed phase shift θ . We suppose for simplicity that $\langle\hat{a}^\dagger\hat{b}\hat{N}^+\rangle = |\langle\hat{a}^\dagger\hat{b}\hat{N}^+\rangle|$, so that the direction of the mean spin of the state to be used in the interferometer will be along the X axis: i.e. when $\phi = 0$: $\langle\hat{J}^X\hat{N}^+\rangle = |\langle\hat{a}^\dagger\hat{b}\hat{N}^+\rangle|$, $\langle\hat{J}^Y\hat{N}^+\rangle = \langle\hat{J}^Z\hat{N}^+\rangle = 0$. For a controlled reference phase shift θ' , two successive orthogonal measurement settings, θ' and $\theta' + \pi/2$, will allow a determination of the unknown phase ϕ :

$$\begin{aligned} \langle\tilde{J}(\phi, \theta')\rangle &= \cos(\phi - \theta') |\langle\hat{a}^\dagger\hat{b}\hat{N}^+\rangle|, \\ \langle\tilde{J}(\phi, \theta' + \pi/2)\rangle &= -\sin(\phi - \theta') |\langle\hat{a}^\dagger\hat{b}\hat{N}^+\rangle|. \end{aligned} \quad (31)$$

A single measurement setting θ' cannot determine the unknown phase completely, since the information given is about $\cos\varphi$ only. The mean differential signal for measurement $\tilde{J}(\phi, \theta') = \hat{J}^\phi\hat{N}^+$ is $-\langle\hat{J}^X\hat{N}^+\rangle\sin\varphi$, and $\Delta\phi$ as given by (30) for this measurement is ($\varphi = \phi - \theta'$)

$$(\Delta\phi)^2 = \left\{ (\Delta\tilde{J}^X)^2 \cot^2(\varphi) + (\Delta\tilde{J}^Y)^2 \right\} / |\langle\tilde{J}^X\rangle|^2, \quad (32)$$

together with a similar expression obtained in the orthogonal direction. The objective is to determine the conditions on the interferometric state so that the uncertainty in the phase estimation is minimized.

The Standard Quantum Limit (SQL) sensitivity $\Delta\phi = 1/\sqrt{N}$, as given by equation (10), is obtained when fields a and b are formed from a number state $|N\rangle$ incident at one port of a beam splitter, with a vacuum state input at the second port [5, 21, 41]. An entangled state results [49], for which $\langle\hat{J}^X\rangle = N/2$, $(\Delta\hat{J}^X)^2 = 0$ and $(\Delta\hat{J}^Y)^2 = N/4$. Then, for all phases φ , it is readily shown that $\Delta\phi \sim 1/\sqrt{N}$.

For some entangled states, it is well-known [40, 50] that the phase sensitivity can be enhanced below the SQL. The most well-studied cases however consider a small phase shift about a fixed phase reference [3, 41]. It is evident from (32) that the maximum differential for $\tilde{J}(\phi, \theta')$ is at $\varphi = \pi/2$, for which $(\Delta\phi)_{\pi/2} = \Delta\tilde{J}^Y / |\langle\tilde{J}^X\rangle|$. The sensitivity at this point is thus given by the normalized spin squeezing parameter (29) which reduces to (19) for fixed number inputs. Sub-shot noise sensitivity is achieved when $\Delta\phi < 1/\sqrt{\langle\hat{N}\rangle}$, so by the definition

(29), sub-shot noise enhancement occurs for interferometric states satisfying

$$\xi_{S,norm} < 1. \quad (33)$$

The technique relies on an accurate estimate ϕ_X of the unknown phase, combined with setting θ to $\phi_X - \pi/2$, so that subsequent measurements measure small shifts near the optimal phase. We will show in the next section that a near Heisenberg-limited sensitivity of $(\Delta\phi)_{\pi/2} \sim O(\sqrt{2}/N)$ is predicted for this case, when the two-modes \hat{a} and \hat{b} of an atom interferometer are prepared from a two-mode double-potential well BEC ground state.

D. Estimation of an unknown phase

The question of phase estimation with an unknown phase and limited number of measurements is a different issue [29]. Where we restrict to phase estimation via the interferometric scheme figure 1 based on the number difference measurements (17-18), we see from (32) that a noise-reduction enhancement over a range of angles, with a reduced variance in *both* $\Delta\hat{J}^X$ and $\Delta\hat{J}^Y$ is needed. This is essential where there is no prior knowledge of the phase ϕ and successive adaptive phase measurements [30] are not possible. We note that the sensitivity of the measurement $\tilde{J}(\phi, \theta')$ is destroyed by the divergent contribution evident in (32) at $\varphi \sim 0, \pi$, unless $\Delta\hat{J}^X = 0$, which places a severe limit on the interferometric state. This is not a necessary consideration, however, if the full phase is to be measured via both orthogonal measurements settings given by (31). The settings in the “quiet” quadrants $\theta' = (\phi + \pi/2) \pm \pi/4$ and $\theta' = (\phi - \pi/2) \pm \pi/4$ have enhanced sensitivity over those in the “noisy” quadrants $\theta' = \phi \pm \pi/4$ and $\theta' = (\phi + \pi) \pm \pi/4$, and for any unknown phase ϕ one of the orthogonal settings θ' or $\theta' + \pi/2$ will be in the useful quadrants. Least squares estimation is an obvious strategy here.

We thus consider the following strategy. The first reading of the pair $\tilde{J}(\phi, 0)$ or $\tilde{J}(\phi, \pi/2)$ determines values for $\cos\phi$ and $\sin\phi$, and thus the location of the phase in the plane. In this way, it can be determined which of $\tilde{J}(\phi, 0)$ or $\tilde{J}(\phi, \pi/2)$ has measured in the quiet quadrants. Sub-shot noise sensitivity is then guaranteed *at all unknown angles* ϕ , for this preferred measurement, provided it can be shown that $\Delta\phi < 1/\sqrt{\langle\hat{N}\rangle}$ across the entire two quiet quadrants. According to (32), the worse-case sensitivity for these quiet quadrants is at $\varphi = \pm\pi/4, \pm3\pi/4$, and is given by $(\Delta\phi)_{max}^2 = [\Delta^2(\tilde{J}^X) + \Delta^2(\tilde{J}^Y)] / |\langle\tilde{J}^X\rangle|^2$. The condition for $(\Delta\phi)_{max}^2$ to be sub-shot noise is $(\Delta\phi)_w < 1/\sqrt{\langle\hat{N}\rangle}$ which is quantified by a phase-sensitivity measure:

$$\eta_{ph} = \frac{\sqrt{\langle\hat{N}\rangle\langle\hat{N}^+\rangle E_{ph}/2}}{|\langle\tilde{J}^X\rangle|} < 1. \quad (34)$$

When the interferometric fields satisfy (34), sub-shot noise sensitivity for all angles ϕ is guaranteed, for any fixed measurement setting θ within the two quiet quadrants for measurement of ϕ .

The fundamental quantum limit for (34) is given by the smallness of η_{ph} , which is linked to the uncertainty relation for the sum of the two spins J^X and J^Y . It is therefore important to determine a tight lower bound on this sum, in order to obtain the ultimate phase interferometric sensitivity. The real question becomes to what extent can we still minimize $\Delta^2 \tilde{J}^Y$, given that the sum $\Delta^2(\tilde{J}^X) + \Delta^2(\tilde{J}^Y)$ is also to be minimized. The answer is not the same as for two complementary observables like two optical quadratures, or position and momentum, for which the commutator is a constant. The uncertainty relation for spin operators has a state-dependent form,

$$\Delta J^X \Delta J^Y \geq |\langle J^Z \rangle|/2.$$

which means the two variances ΔJ^X and ΔJ^Y can both be reduced below the shot noise level. For $\langle J^Z \rangle = 0$ the Heisenberg uncertainty principle is unable to give any bound at all. This is possibly misleading, since the Heisenberg uncertainty principle is simply a bound that does not guarantee it can be saturated. As discussed in section IIID, a recent analysis of the spin-variance uncertainty leads to a tight bound on the variances:

$$\Delta^2 J^X + \Delta^2 J^Y \geq C_J,$$

where C_J is a function of the total spin J with an asymptotic limit of $3(2J)^{2/3}/8$. For $J = N/2$ one finds $\Delta\phi \geq \sqrt{C_J}/J \rightarrow O(\sqrt{1.5}/N^{2/3})$, which we will show in the next section is predicted for the two-well BEC ground state, in the both attractive and repulsive regimes. The fundamental limit is below the SQL of $O(1/\sqrt{N})$ over all the phase angles in the half-plane, but can only reach the Heisenberg limit of $O(1/N)$ over part of the range. We next present the details of how these levels of sensitivity can be realized in a BEC interferometer.

V. BEC INTERFEROMETER

We consider how the criteria (28), (29) and (34) for entanglement and phase-measurement sensitivity are satisfied in a typical cold-atom experiment. The mechanism for two-mode squeezing and entanglement here is similar to that first realized for optical modes using four-wave mixing [51], except employing the ground-state of an interacting BEC. We consider an idealized two-mode or two-well BEC with a normalized self-interaction coefficient g , and a linear tunneling coupling κ between two modes with boson operators \hat{a} and \hat{b} , described by the Hamiltonian [52]

$$H = \kappa(a^\dagger b + ab^\dagger) + \frac{g}{2}[a^\dagger a^\dagger aa + b^\dagger b^\dagger bb]. \quad (35)$$

Solutions for the two-mode entanglement of the ground state have been presented in Refs. [9, 49], for the case of an initial state of N atoms distributed evenly between the wells.

In order to interpret these figures it should be kept in mind that the attractive BEC case generates a nearly perfect planar quantum-squeezed (PQS) state. This is close to an optimal minimization of the variance sum, reducing phase-noise in both quadratures. The resulting state is used in direct interferometry, with a single beam-splitter. On the other hand, while the repulsive BEC case also produces a PQS state, it has a different characteristic with one of the quadratures having a phase-noise level reduced almost to the Heisenberg limit, while the other phase quadrature is not at the Heisenberg limit. This state is used in Mach-Zehnder interferometry, with an additional phase rotation to generate the state with the optimal characteristics for phase-measurement.

A. Entanglement criteria

We present solutions for the ground state of (35), including number fluctuations as described in section IV A. The dashed curves in figure 2 show that the normalized phase-entanglement criteria Eq. (28) $E_{ph} < 1$ detects the two-mode entanglement of the ground state of a two-well BEC [4–6, 9] in a way that is almost immune to Poissonian number fluctuations. On the other hand, the solid curves for E_{HZ} reveal that the entanglement detected via the un-normalized Hillery-Zubairy criterion $E_{HZ} < 1$ is very easily destroyed by number fluctuations.

The figure 3 shows that the normalized spin squeezing parameter also detects squeezing and particle entanglement in a way that is insensitive to number fluctuations. The solid curves plot the squeezing predicted for a fixed $N = 100$, where we recall that this parameter is not defined except in the case of fixed N , but that according to Eq. (19) will detect both squeezing and entanglement among the N particles when $\xi_S < 1$. The dashed curves plot the results for squeezing of the normalized parameter, which requires $\xi_{S,norm} < 1$ for detection of entanglement but in the presence of arbitrary number N , to show perfect overlay. Our proof justifies the normalization procedure used in recent experiments that report spin squeezing and particle entanglement [4–6], for a repulsive BEC with fluctuating total numbers.

The criteria Eq. (28) and (29) enable an unambiguous detection of entanglement in the presence of number fluctuations. We note that a similar immunity of the Peres positive partial transpose PPT entanglement measure to loss was shown in Ref. [53], and for other entanglement measures to loss in Ref [54].

The figures include both attractive ($g < 0$) and repulsive ($g > 0$) interactions. In the attractive case, the criteria (23-28) for entanglement are satisfied when applied directly to the modes of the two wells, then described by a, b . In the repulsive case, an interferometric

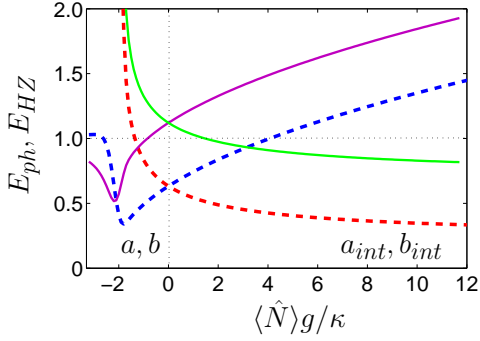


Figure 2. (Color online) Entanglement of a ground state BEC including Poissonian atom number fluctuations. Here $N = 100$, for fixed $g/\kappa = 10^3$. (i) Entanglement between wells a and b is detected if $E_{ph} < 1$ (blue and red dashed curves) or $E_{HZ} < 1$ (equation (23) (purple and green solid curves). Curves a, b (purple solid and blue dashed) are for two-well BEC modes a, b ; curves a_{int}, b_{int} (green solid and red dashed) are for a, b formed from BEC modes a_i, b_i input to the M-Z interferometer sequence depicted in Figure 1.

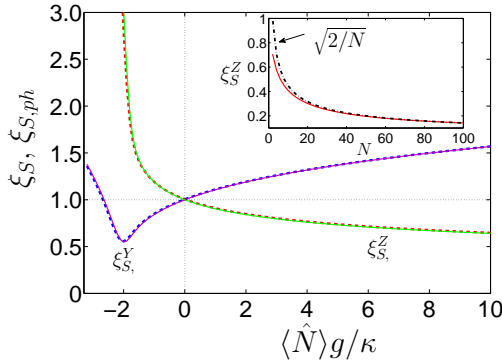


Figure 3. (Color online) Spin squeezing and phase measurement sensitivity parameters for ground state of a two-well BEC with $\langle \hat{N} \rangle = 100$. Plot of spin squeezing parameters ξ_S^Y, ξ_S^Z (purple and green solid curves) and the normalized parameters $\xi_{S,ph}^Y, \xi_{S,ph}^Z$ (blue and red dashed curves), for Poissonian number fluctuations. States with $\xi_S^{Y/Z}, \xi_{S,ph}^{Y/Z} < 1$ show sub-shot noise enhanced phase sensitivity for measurements of small rotations about a fixed phase. Inset shows the asymptotic $\sqrt{2/N}$ ($(\Delta\phi)_{\pi/2} \sim O(\sqrt{2/N})$) behavior with increasing N for fixed $g/\kappa = 10^3$. The Heisenberg limit is $\xi_S^{Y/Z} \sim O(\sqrt{1/N})$ [32].

sequence is necessary: two-well modes a_i, b_i are phase-shifted and placed through a MZ beam-splitter. The reason for this is that while an attractive BEC reduces fluctuations directly in the plane of $\hat{J}^{X,Y}$, a repulsive BEC reduces fluctuations in a different plane, namely in $\hat{J}^{X,Z}$. Without the additional input phase-shifter, the phase-measurement is in the $Y-Z$ plane, where only one phase has reduced fluctuations, as observed [5].

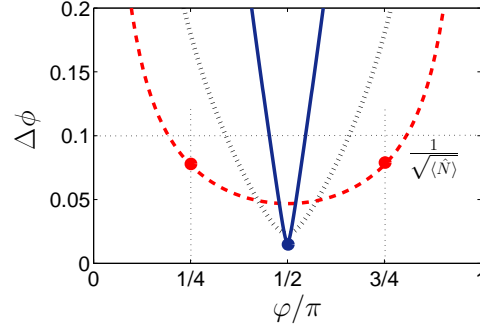


Figure 4. (Color online) Measured phase uncertainty $\Delta\phi$ (32), for Poissonian fluctuations. Phase sensitivity is better than shot noise level if $\Delta\phi < 0.1 = 1/\sqrt{\langle \hat{N} \rangle}$. The red dashed curve corresponds to the critical value of $\langle \hat{N} \rangle g/\kappa \simeq 43.6$ which gives the lowest value of E_{ph} for the repulsive regime. Increasing $\langle \hat{N} \rangle g/\kappa$ to 10^3 (dotted) and 10^4 (solid), improves the optimum phase uncertainty, but the useful region of phase angles become narrower. Optimum sensitivity is at $\varphi \pm \pi/2$ and is determined by the spin squeezing parameter $\xi_{S,norm}$. For detecting an unknown phase using only two orthogonal measurements, the sensitivity over the quiet quadrant $\varphi/\pi = \frac{1}{2} \pm \frac{1}{4}$ becomes relevant.

B. BEC interferometric phase measurement

Now we turn to the question of using the BEC two-mode states for the purpose of interferometric phase measurement. The spin squeezing measure ξ_S in the presence of fixed N , and the normalized parameter $\xi_{S,norm}$ in the presence of fluctuating numbers, give sensitivity for measurements of small rotations about a fixed phase below the SQL when $\xi_S < 1$ and $\xi_{S,norm} < 1$ (Eq. (33)). The figure 3 reveals this spin squeezing to be predicted for a wide range of parameters of the ground state solution. The inset shows the reduction in noise to be near Heisenberg limited with the scaling $\xi_S^{Y/Z} \sim O(\sqrt{2/N})$ evident.

The worse-case sensitivity of the interferometer to an arbitrary angle defined within the two quiet quadrants of measurement for an unknown phase ϕ is given by the η_{ph} of Eq. (34), which is minimized according to the phase entanglement measure E_{ph} of figure 4. The best scaling of E_{ph} with N is given as $J^{2/3}$, that of the C_J coefficients, and is achieved at the critical value of Ng/κ . This implies, from the phase sensitivity measure η_{ph} , a sensitivity of $(\Delta\phi)_{worse} \sim O(N^{-2/3})$.

For the proposed BEC interferometer, one can evaluate the actual range of sensitivities for the unknown incoming phase ϕ using Eq (32) directly. The different range of phase-noise reduction as a function of measured phase-angle and BEC interaction strength is shown in figure 4. The best sensitivity is obtained at $\varphi = \pi/2$ and the value for $\Delta\phi$ is determined by the spin squeezing parameter Eq. (29) $\xi_{S,norm}$, which reduces to ξ_S for fixed number N . Where one measures an unknown phase ϕ using only two orthogonal measurements ($\theta = 0$ and $\theta' = \pi/2$), only the sensitivity over the “quiet” quadrant indicated on the graph by the region $\varphi/\pi = 1/2 \pm 1/4$ becomes relevant.

In this case, the worse case sensitivities are at the edges ($\varphi = \pi/4$ and $\varphi = 3\pi/4$) and determined by the value of η_{ph} , Eq. (33). This parameter is optimized by minimizing the two-mode entanglement parameter E_{ph} , which reduces to the Hillery Zubairy entanglement parameter when N is fixed. This demonstrates the trade-off between noise-reduction and range of measurable phase.

VI. CONCLUSIONS

In summary, we have introduced the normalized relative phase quadrature operator $\tilde{J}(\phi)$ as the most direct operational expression of how interferometric measurements give rise to phase information. A corresponding phase-entanglement measure, as quantified by E_{ph} , describes a useful physical resource for phase measurement. This directly quantifies the measurement sensitivity increase above the standard quantum limit, as E_{ph} decreases towards a maximally entangled state. We also introduce a normalized phase squeezing measure, $\xi_{S,ph}^{Z/Y}$, which signifies entanglement between qubits or particles. Both measures are normalized in terms of the total particle number, using the generalized Moore-Penrose inverse method. We show how it is possible to improve BEC phase measurements so that a range of unknown phases have reduced phase-noise, rather than just one pre-selected phase.

APPENDIX

In this Appendix, we provide a detailed proof of the normalized entanglement criterion, equation (28) and the normalized spin-squeezing criterion, (29).

Phase-entanglement criterion

We wish to show that entanglement between modes a and b is confirmed via an entanglement phase criterion, equation (28). First, we note the general result that if \hat{N} commutes with an arbitrary hermitian operator \hat{O} having eigenvalues j , we can introduce a limiting procedure to obtain a normalized mean value, $\langle \hat{O} \rangle = \langle \hat{O} \hat{N}^+ \rangle$, where \hat{N}^+ is the generalized Moore-Penrose [35] inverse of the number operator \hat{N} , so that:

$$\begin{aligned} \langle \hat{O} \hat{N}^+ \rangle &= \lim_{\epsilon \rightarrow 0} \sum_{n,j} \frac{nj}{n^2 + \epsilon} P(n, j) \\ &= \sum_{n,j} n^+ j P(n, j). \end{aligned} \quad (36)$$

Here $P(n, j)$ is the probability for simultaneous outcomes n, j for \hat{N} and \hat{O} respectively, and the eigenvalues of the generalized inverse operator \hat{N}^+ are $n^+ = n^{-1}$ for $n > 0$,

with $n^+ = 0$ for $n = 0$. Hence, the expectation value for the ratio becomes:

$$\langle \tilde{O} \rangle = \sum_{n \geq 0} \langle \tilde{O} \rangle_n P_n,$$

where $\langle \hat{O} \rangle_n = \sum_j P(j|n)j$, $P_n = \sum_j P(n, j)$, $P(j|n) = P(n, j)/P_n$, and we define $\langle \tilde{O} \rangle_n = \langle \hat{O} \rangle_n n^+$. Similarly, the corresponding variances, $\Delta^2 \tilde{O} = \langle \tilde{O}^2 \rangle - \langle \tilde{O} \rangle^2$, can be expanded as:

$$\Delta^2 \tilde{O} = \sum_n \langle \tilde{O}^2 \rangle_n P_n - \left[\sum_n \langle \tilde{O} \rangle_n P_n \right]^2.$$

However, we know from elementary variance properties that $\sum_n \langle \tilde{O} \rangle_n^2 P_n \geq \left[\sum_n \langle \tilde{O} \rangle_n P_n \right]^2$, hence we can write that:

$$\begin{aligned} \Delta^2 \tilde{O} &\geq \sum_n \langle \tilde{O}^2 \rangle_n P_n - \sum_n \langle \tilde{O} \rangle_n^2 P_n \\ &= \sum_n \left[\langle \tilde{O}^2 \rangle_n - \langle \tilde{O} \rangle_n^2 \right] P_n \\ &= \sum_n \left[\Delta_n^2 \hat{O} \right] (n^+)^2 P_n. \end{aligned} \quad (37)$$

Here, as usual, we have defined $\Delta_n^2 \hat{O} = \sum_j P(j|n)j^2 - \langle \hat{O} \rangle_n^2$.

Next, we apply this result to normalized spin variances, giving:

$$(\Delta \tilde{J}^X)^2 + (\Delta \tilde{J}^Y)^2 \geq \sum_n (n^+)^2 P_n [\Delta_n^2 \hat{J}^X + \Delta_n^2 \hat{J}^Y], \quad (38)$$

where we have used the definitions that:

$$\Delta^2 \tilde{J}^X = \sum_n P_n \langle \tilde{J}^{X2} \rangle_n - \langle \tilde{J}^X \rangle^2.$$

Finally, if we assume the separability equation (22), we see that for a separable density matrix with a fixed total particle number:

$$\Delta_n^2 \hat{J}^X + \Delta_n^2 \hat{J}^Y \geq n/2,$$

which means that for normalized operators,

$$\begin{aligned} (\Delta \tilde{J}^X)^2 + (\Delta \tilde{J}^Y)^2 &\geq \frac{1}{2} \sum_n n (n^+)^2 P_n \\ &= \frac{1}{2} \sum_n n^+ P_n \\ &= \frac{1}{2} \langle \hat{N}^+ \rangle. \end{aligned}$$

Here we note that the generalized inverse has many of the properties of the standard inverse, in particular that $\hat{N} (\hat{N}^+)^2 = \hat{N}^+$, and that of course no singularities occur with this criterion. When this condition is violated, we must have an entangled state, which leads to (28).

Phase-squeezing criterion

Next, we wish to show that entanglement between N spin-1/2 systems, where the number of systems can fluctuate, is confirmed by the normalized spin squeezing criterion of equation (29). For N spin 1/2 separable systems, where N is fixed and nonzero, one finds that [38]:

$$(\Delta \hat{J}^Z)^2 \geq \frac{1}{N} [\langle \hat{J}^X \rangle^2 + \langle \hat{J}^Y \rangle^2] \\ \geq N^+ \langle \hat{J}^X \rangle^2.$$

The last expression, using the generalized inverse of N , holds even when $N = 0$. Thus, using the normalized variance inequality (37), we obtain:

$$\Delta^2 \tilde{J}^Z \geq \sum_n P_n (n^+)^2 [\Delta_n^2 \hat{J}^Z], \quad (39)$$

$$\geq \sum_n P_n n^+ \langle \hat{J}^X \hat{N}^+ \rangle_n^2. \quad (40)$$

Therefore, using the result that $n^2 n^+ = n$, we see that:

$$\langle \hat{N} \rangle \Delta^2 \tilde{J}^Z \geq \left\{ \sum_{n>0} P_n n^2 n^+ \right\} \sum_{n>0} P_n n^+ \langle \hat{J}^X \hat{N}^+ \rangle_n^2.$$

Next, we use the Cauchy-Schwarz inequality: $\{\sum_{n>0} x_n^2\} \{\sum_{n>0} y_n^2\} \geq |\sum_{n>0} x_n y_n|^2$ where $x_n = \sqrt{n P_n}$ and $y_n = \sqrt{P_n n^+ \langle \hat{J}^X \hat{N}^+ \rangle_n}$, and hence:

$$\langle \hat{N} \rangle \Delta^2 \tilde{J}^Z \geq \left[\sum_{n>0} P_n \langle \hat{J}^X \hat{N}^+ \rangle_n \right]^2 \\ = |\langle \hat{J}^X \hat{N}^+ \rangle|^2 = |\langle \tilde{J}^X \rangle|^2.$$

This proves the phase-squeezing criterion, equation (29).

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